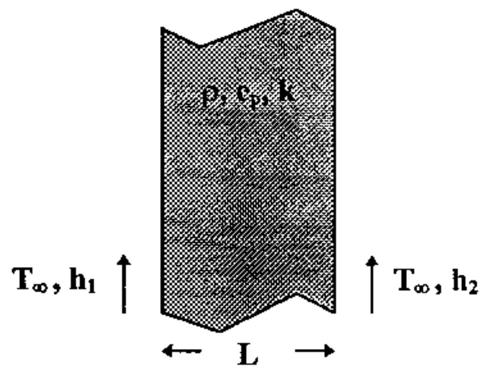
Answer all questions
Open book, notes, and handouts

Name: SOLUTION

25 1. Consider a plane wall, initially at a uniform temperature T_0. At time t=0, both surfaces of the wall are suddenly exposed to convection with a fluid at temperature T_∞. While the convective temperature on each side of the wall is the same, the convection heat transfer coefficients are different, as shown in the Figure.



10 a) Write the differential equation, boundary conditions, and initial conditions for this problem using dimensional variable (e.g., temperature in °C, length in meters, etc.)

b) Show that, by proper selection of normalized variables, this problem can be expressed as a problem with linear and homogeneous partial differential equation and boundary conditions.

p.d.e. $\quad \frac{\partial^2 T(x^*,t)}{\partial x^{*2}} = \frac{1}{\alpha}\frac{\partial T(x^*,t)}{\partial t} \qquad x^* [=] m$

b.c. ① $q(0,t) = h_1[T_\infty - T(0,t)] = -k \left.\frac{\partial T(x^*,t)}{\partial x^*}\right|_{x^*=0}$

② $q(L,t) = h_2[T(L,t) - T_\infty] = -k \left.\frac{\partial T(x^*,t)}{\partial x^*}\right|_{x^*=L}$

i.c. $T(x^*,0) = T_0$

Define: $u = \frac{T-T_\infty}{T_0-T_\infty} \qquad x = \frac{x^*}{L} \qquad \theta = \frac{\alpha t}{L^2}$

$$\frac{\partial T}{\partial x^*} = \frac{1}{L}\frac{\partial T}{\partial x} \qquad \frac{\partial^2 T}{\partial x^{*2}} = \frac{1}{L^2}\frac{\partial T}{\partial x^2} \qquad \frac{\partial T}{\partial t} = \frac{\alpha}{L^2}\frac{\partial T}{\partial \theta}$$

$$\partial T = (T_0 - T_\infty)\,\partial u$$

p.d.e. $\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \theta}$ ← homogeneous, linear

b.c. ① $\left.\frac{\partial u}{\partial x}\right|_{x=0} = Bi_1\, u(0,\theta) \qquad Bi_1 = \frac{h_1 L}{k}$

② $\left.\frac{\partial u}{\partial x}\right|_{x=1} = -Bi_2\, u(1,\theta) \qquad Bi_2 = \frac{h_2 L}{k}$

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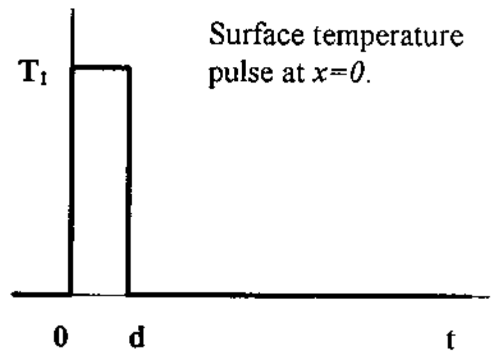
2. The temperature distribution in a plane wall, initially at $T=T_0$, is exposed to a step change at time $t=0$ in surface temperature at $x=0$ to $T=T_1$, can be described by the following equation:

$$U(x,t) = \frac{T-T_0}{T_1-T_0} = 1 - \frac{x}{L} - 2\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)e^{-n^2\pi^2\alpha t/L^2}}{n\pi} \quad \textbf{(1)}$$

Using superposition, develop an equation for the temperature response for $t>d$ to a rectangular pulse in surface temperature at $x=0$, shown in the Figure.



Surface temperature pulse at $x=0$.

$t>d$

$$\frac{T-T_0}{T_1-T_0} = U(x,t) - U(x,t-d)$$

$$= 1 - x/L - 2\sum \frac{\sin n\pi x/L}{n\pi} e^{-n^2\pi^2\alpha t/L^2}$$

$$-\left(1 - x/L - 2\sum \frac{\sin n\pi x/L}{n\pi} e^{-n^2\pi^2\alpha (t-d)/L^2}\right)$$

$$= 2\sum \frac{\sin n\pi x/L}{n\pi}\left(e^{-n^2\pi^2\alpha(t-d)/L^2} - e^{-n^2\pi^2\alpha t/L^2}\right)$$

or

$$= 2\sum \frac{\sin n\pi x/L}{n\pi} e^{-n^2\pi^2\alpha t/L^2}\left[e^{n^2\pi^2\alpha d/L^2} - 1\right]$$

3. Consider a system that can be described by the following first-order ordinary differential equation with constant coefficients.

$$\frac{du}{dt} + au = g(t) \qquad (2)$$

Initial condition: $u(0) = 0$

20 a) For $g(t) = b$, with b a constant, use Laplace transforms to show that the solution to Equation 2 is

$$u(t) = \frac{b}{a}\left(1 - e^{-at}\right) \qquad (3)$$

$$\mathcal{L}[u'] = s\hat{u}(s)$$

$$s\hat{u}(s) + a\,\hat{u}(s) = \hat{g}(s)$$

$$\hat{g}(s) = \mathcal{L}[b] = b/s$$

$$\hat{u}(s) = \frac{b}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$$

$$b = A(s+a) + Bs$$

$$b = (A+B)s + Aa$$

$$A = \frac{b}{a}$$

$$B = -A = -b/a$$

$$\hat{u}(s) = \frac{b}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right]$$

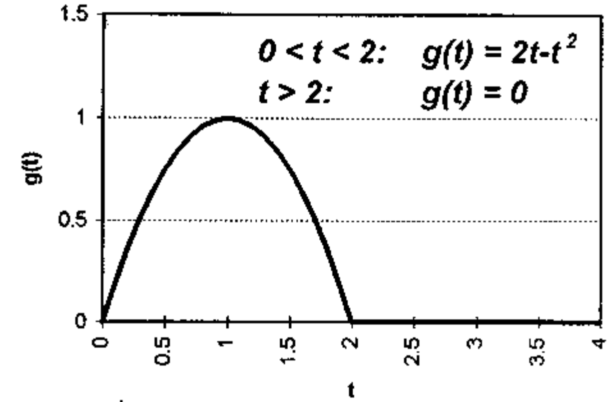
$$\mathcal{L}^{-1}[1/s] = 1$$

$$\mathcal{L}^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$u(t) = \frac{b}{a}\left[1 - e^{-at}\right]$$

30 b) Duhamel's theorem can be used with Equation 1 to solve for an arbitrary input, or forcing function, *g(t)*. In other words, the fundamental solution of Equation 2, *U(t)*, can be obtained by setting *g(t) = 1*. The solution for an arbitrary input *g(t)* can then be developed using Duhamel's theorem.

Consider the system with *g(t)* shown in the Figure. Using the attached integral tables, develop the solution to the system using the fundamental solution of Equation 3 with *b = 1* and Duhamel's theorem.



$$U(t) = \frac{1}{a}\left(1-e^{-at}\right)$$

<u>t<2</u>

$$u(t) = \int_0^t U(t-\tau)\frac{\partial g(\tau)}{\partial \tau}\,d\tau$$

$$= \int_0^t \frac{1}{a}\left(1-e^{-a(t-\tau)}\right)(2-2\tau)\,d\tau$$

$$= \frac{2}{a}\int_0^t \left[1 - e^{-at}e^{a\tau} - \tau + \tau e^{-at}e^{a\tau}\right]d\tau$$

$$= \frac{2}{a}\left[t - e^{-at}\frac{1}{a}\left(e^{at}-1\right) - \frac{t^2}{2} + e^{-at}\int_0^t \tau e^{a\tau}\,d\tau\right]$$

$$\int_0^t \tau e^{a\tau}\,d\tau = e^{a\tau}\left[\frac{\tau}{a} - \frac{1}{a^2}\right]_{\tau=0}^{t}$$

$$= e^{at}\left[\frac{t}{a} - \frac{1}{a^2}\right] + \frac{1}{a^2} = e^{at}\left[t/a - \frac{1}{a^2}\left(1-e^{-at}\right)\right]$$

$$= \frac{2}{a}\left[t - \frac{1}{a}\left(1-e^{-at}\right) - \frac{t^2}{2} + t/a - \frac{1}{a^2}\left(1-e^{-at}\right)\right]$$

t>2

$$u(t) = \frac{2}{a}\int_0^2 \left[1-e^{-at}e^{a\tau} - \tau + \tau e^{-at}e^{a\tau}\right]d\tau$$

$$= \frac{2}{a}\left[2 - e^{-at}\frac{1}{a}\left(e^{2a}-1\right) - \frac{4}{2} + e^{-at}e^{2a}\left[\frac{2}{a} - \frac{1}{a^2}\left(1-e^{-2a}\right)\right]\right]$$

567.1. $\int x e^{ax} dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right].$

567.2. $\int x^2 e^{ax} dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right].$

567.3. $\int x^3 e^{ax} dx = e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right].$

567.8. $\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax} dx.$

567.9. $\int x^n e^{ax} dx = e^{ax}\left[\frac{x^n}{a} - \frac{n x^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \cdots + (-1)^{n-1}\frac{n!x}{a^n} + (-1)^n \frac{n!}{a^{n+1}}\right], \qquad [n \geqq 0].$

568.1. $\int \frac{e^{ax}dx}{x} = \log|x| + \frac{ax}{1!} + \frac{a^2x^2}{2\cdot 2!} + \frac{a^3x^3}{3\cdot 3!} + \cdots + \frac{a^n x^n}{n\cdot n!} + \cdots, \qquad [x^2 < \infty].$

568.11. For $\int \frac{c^x dx}{x}$, note that $c^x = e^{x \log c}$.

568.2. $\int \frac{e^{ax}dx}{x^2} = -\frac{e^{ax}}{x} + a\int \frac{e^{ax}dx}{x}.$ [See **568.1.**]

568.3. $\int \frac{e^{ax}dx}{x^3} = -\frac{e^{ax}}{2x^2} - \frac{ae^{ax}}{2x} + \frac{a^2}{2}\int \frac{e^{ax}dx}{x}.$ [See **568.1.**]

568.8. $\int \frac{e^{ax}dx}{x^n} = -\frac{e^{ax}}{(n-1)x^{n-1}} + \frac{a}{n-1}\int \frac{e^{ax}dx}{x^{n-1}}, \qquad [n > 1].$

568.9. $\int \frac{e^{ax}dx}{x^n} = -\frac{e^{ax}}{(n-1)x^{n-1}} - \frac{ae^{ax}}{(n-1)(n-2)x^{n-2}} - \cdots - \frac{a^{n-2}e^{ax}}{(n-1)!x} + \frac{a^{n-1}}{(n-1)!}\int \frac{e^{ax}dx}{x},$

$[n > 1]$. [See **568.1.**]

569. $\int \frac{dx}{1+e^x} = x - \log(1+e^x) = \log\frac{e^x}{1+e^x}.$

569.1. $\int \frac{dx}{a + be^{px}} = \frac{x}{a} - \frac{1}{ap}\log|a + be^{px}|.$

570. $\int \frac{xe^x dx}{(1+x)^2} = \frac{e^x}{1+x}.$ **570.1.** $\int \frac{xe^{ax}dx}{(1+ax)^2} = \frac{e^{ax}}{a^2(1+ax)}.$

575.1. $\int e^{ax}\sin x\, dx = \frac{e^{ax}}{a^2+1}(a\sin x - \cos x).$

575.2. $\int e^{ax}\sin^2 x\, dx = \frac{e^{ax}}{a^2+4}\left(a\sin^2 x - 2\sin x\cos x + \frac{2}{a}\right).$

575.3. $\int e^{ax}\sin^3 x\, dx = \frac{e^{ax}}{a^2+9}\left[a\sin^3 x - 3\sin^2 x\cos x + \frac{6(a\sin x - \cos x)}{a^2+1}\right].$

575.9. $\int e^{ax}\sin^n x\, dx = \frac{e^{ax}\sin^{n-1}x}{a^2+n^2}(a\sin x - n\cos x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\sin^{n-2} x\, dx.$

576.1. $\int e^{ax}\cos x\, dx = \frac{e^{ax}}{a^2+1}(a\cos x + \sin x).$

576.2. $\int e^{ax}\cos^2 x\, dx = \frac{e^{ax}}{a^2+4}\left(a\cos^2 x + 2\sin x\cos x + \frac{2}{a}\right).$

576.3. $\int e^{ax}\cos^3 x\, dx = \frac{e^{ax}}{a^2+9}\left[a\cos^3 x + 3\sin x\cos^2 x + \frac{6(a\cos x + \sin x)}{a^2+1}\right].$

576.9. $\int e^{ax}\cos^n x\, dx = \frac{e^{ax}\cos^{n-1}x}{a^2+n^2}(a\cos x + n\sin x) + \frac{n(n-1)}{a^2+n^2}\int e^{ax}\cos^{n-2} x\, dx.$ [Ref. 2, p. 141.]

577.1. $\int e^{ax}\sin nx\, dx = \frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx).$

577.2. $\int e^{ax}\cos nx\, dx = \frac{e^{ax}}{a^2+n^2}(a\cos nx + n\sin nx).$

[Ref. 7, p. 9.]